

STRONG APPROXIMATION FOR THE VARIETY CONTAINING A TORUS

DASHENG WEI

ABSTRACT. Let X be a smooth and geometrically integral variety over a number field k . Suppose X contains a torus and $\bar{k}[X]^\times = \bar{k}^\times$, let $W \subset X$ be a closed subset of codimension at least 2. In this note, we proved that $X \setminus W$ satisfies strong approximation with étale algebraic Brauer–Manin obstruction off one place. Furthermore, if $\text{Pic}(\bar{X})$ is torsion free, then $X \setminus W$ satisfies strong approximation with algebraic Brauer–Manin obstruction off one place; if $\text{Pic}(\bar{X})$ is not torsion free, a counterexample was given that does not satisfy strong approximation with Brauer–Manin obstruction off one place. On the other hand, if X is a toric variety (maybe $\text{Pic}(\bar{X})$ is not torsion free), we also proved that $X \setminus W$ satisfies strong approximation with algebraic Brauer–Manin obstruction off one place.

0. INTRODUCTION

Let X be a variety over a number field k and S a finite set of places of k . We say that strong approximation holds for X off S if the image of the set $X(k)$ of rational points is dense in the space $X(\mathbb{A}_k^S)$ of adelic points on X outside S . Strong approximation for X off S implies the Hasse principle for S -integral points on any S -integral model of X .

For a proper variety X , $X(\mathbb{A}_k^S) = \prod_{v \notin S} X(k_v)$, and the adelic topology coincides with the product topology. A proper variety satisfies strong approximation off S if and only if weak approximation for the rational points holds off S . For an affine variety X , however, studying strong approximation seems to be generally much harder than studying weak approximation for its proper models.

For algebraic groups and their homogeneous spaces, weak and strong approximation have been widely studied. For certain simply connected semisimple groups and their principal homogeneous spaces, we have the classical work of Kneser, Harder, Platonov and others. For certain homogeneous spaces of connected algebraic groups with connected stabilizers, see for example [Bor96, CTX09, Har08, Der11, WX12, WX13, BD13] and the references therein for weak and strong approximation with Brauer–Manin obstruction.

Few strong approximation results are known for more general varieties which are not homogeneous spaces. For strong approximation with Brauer–Manin obstruction for affine varieties defined by equations of the form

$$P(t) = q(z_1, z_2, z_3),$$

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see [CTX13]. For more general fibrations over \mathbb{A}_k^1 with split (e.g., geometrically integral) fibers, see [CTH12]. For some affine varieties defined by equations of the form

$$P(\mathbf{t}) = N_{K/k}(\mathbf{z}),$$

here $P(\mathbf{t}) \in k[t_1, \dots, t_s]$ is a polynomial over k and $N_{K/k}$ is a norm form for an extension K/k , see [DW13].

Define

$$X(\mathbf{A}_k)^{\text{ét}, \text{Br}_1} := \bigcap_{\substack{f: Y \xrightarrow{G} X \\ G \text{ finite}}} \bigcup_{[\sigma] \in H^1(k, G)} f^\sigma(Y^\sigma(\mathbf{A}_k)^{\text{Br}_1}).$$

We say that *strong approximation with (étale) algebraic Brauer–Manin obstruction holds for X off S* if $X(k)$ is dense in the image of $X(\mathbf{A}_k)^{\text{Br}_1}$ (resp. $X(\mathbf{A}_k)^{\text{ét}, \text{Br}_1}$) in $X(\mathbf{A}_k^S)$ under the natural projection.

All notation is standard, we refer it to [CTX13]. We study strong approximation with Brauer–Manin obstruction for a variety which contains a torus. More precisely, we have the following two results.

Theorem 0.1. *Let X be a smooth and geometrically integral variety over k with $\bar{k}[X]^\times = \bar{k}^\times$. Suppose X contains an open subset which is a principal homogeneous space of a k -torus. Let $W \subset X$ be a closed subset of codimension at least 2. Then $X \setminus W$ satisfies strong approximation with étale algebraic Brauer–Manin obstruction off v_0 , where v_0 is a place of k . Furthermore, if $\text{Pic}(\bar{X})$ is torsion free, then $X \setminus W$ satisfies strong approximation with algebraic Brauer–Manin obstruction off v_0 .*

If $\text{Pic}(\bar{X})$ is not torsion free, generally X does not satisfy strong approximation with Brauer–Manin obstruction off v_0 (see Example 1.6). To our knowledge, Theorem 0.1 gives the first family of varieties which satisfy strong approximation with étale algebraic Brauer–Manin obstruction but do not satisfy strong approximation with Brauer–Manin obstruction.

For the proof of Theorem 0.1, if $\text{Pic}(\bar{X})$ is torsion free, we use descent theory (see [CTS87]) to reduce the problem to strong approximation on their universal torsors. By the local description of the universal torsor, there is a quasi-split torus in it. The quasi-split torus can naturally embed into an affine space, this embedding can be extended to a map from an open subset of the affine space to the universal torsor such that the complement of the open subset has codimension at least 2, which ensures strong approximation holds for the universal torsor. If $\text{Pic}(\bar{X})$ is not torsion free, we discover the fact that torsors of X under the maximal torsion subgroup of $\text{Pic}(\bar{X})$ are just the varieties in Theorem 0.1 and their Picard groups are torsion free, this implies that strong approximation with étale algebraic Brauer–Manin obstruction holds on X .

Theorem 0.2. *Let X be a smooth toric variety over k with $\bar{k}[X]^\times = \bar{k}^\times$. Let $W \subset X$ be a closed subset of codimension at least 2. Then $X \setminus W$ satisfies strong approximation with algebraic Brauer–Manin obstruction off v_0 , where v_0 is a place of k .*

A similar result to Theorem 0.2 was also proved by Cao and Xu in [CX14]. In fact, they proved any toric variety X satisfies strong approximation with Brauer–Manin obstruction off all infinite places (without $\bar{k}[X]^\times = \bar{k}^\times$). If $\bar{k}[X]^\times \neq \bar{k}^\times$, generally $X \setminus W$ does not satisfy strong approximation with Brauer–Manin obstruction off infinite places (see [CX14, Example 5.2]). Therefore our result can not be generalized to their case.

In Section 3, we apply Theorem 0.1 to some varieties defined by one multi-norm equation, the computation of Brauer groups is also given in this section.

1. THE PROOF OF THEOREM 0.1

For a number field k , fix an algebraic closure \bar{k} , and let Γ_k be the absolute Galois group. In this section, we mainly prove Theorem 0.1 by descent theory (see [CTS87]).

Lemma 1.1. *Let Z be a closed subset of \mathbb{A}_k^n of codimension at least 2. Then $Y := \mathbb{A}_k^n \setminus Z$ satisfies strong approximation off v_0 , where v_0 is a place of k .*

Proof. Let \mathfrak{Y} be an integral model of Y . For any finite set $S \subset \Omega_k \setminus \{v_0\}$ containing $\infty_k \setminus \{v_0\}$ and any

$$(p_v) \in \prod_{v \in S \cup \{v_0\}} Y(k_v) \times \left(\prod_{v \notin S \cup \{v_0\}} \mathfrak{Y}(\mathbb{O}_v) \right),$$

we must find $p \in Y(k)$ arbitrarily close to p_v for all $v \in S$ with $p \in \mathfrak{Y}(\mathbb{O}_v)$ for all $v \notin S \cup \{v_0\}$.

We will deduce Y to a line ($\cong_k \mathbb{A}_k^1$) in Y . It is enough to find a line $\ell \subset Y$ such that

- (1) $\ell(k_v)$ contains a point p'_v very close to p_v for all $v \in S$, and
- (2) $\ell(k_v) \cap \mathfrak{Y}(\mathbb{O}_v) \neq \emptyset$ for all $v \notin S \cup \{v_0\}$.

Then the proof follows from that $\ell \cong_k \mathbb{A}_k^1$ satisfies strong approximation off v_0 .

For this, we construct a suitable line ℓ . We can choose a point $M \in Y(k)$ arbitrarily close to p_v for $v \in S \cup \{v_0\}$ because \mathbb{A}_k^n satisfies weak approximation. There is a finite set $S' \subset \Omega_k \setminus (S \cup \{v_0\})$ such that, for all $v \notin S' \cup S \cup \{v_0\}$, we have $M \in \mathfrak{Y}(\mathbb{O}_v)$. This gives (1),(2) outside S' for any line through M .

Recall $Z = (\mathbb{A}_k^n \setminus Y)$ has codimension at least 2. The closure Z' of the union of all lines through M meeting Z is a closed subvariety of \mathbb{A}_k^n of codimension at least 1. Therefore, $Y' := (\mathbb{A}_k^n \setminus Z')$ is a dense open subset of \mathbb{A}_k^n . For all $v \in S'$, we choose arbitrary $p_v \in \mathfrak{Y}(\mathbb{O}_v)$. Then we have $N \in Y'(k)$ very close to p_v . This gives (2) for S' for any line through N . Let ℓ be the line through M and N , then $\ell \subset Y$ and satisfies (1) and (2), hence we complete the proof. \square

If $\text{Pic}(\bar{X})$ is torsion free, the following proposition shows that any universal torsors of X as in Theorem 0.1 satisfies strong approximation off one place.

Proposition 1.2. *Let Y be a smooth and geometrically integral variety over k with $\bar{k}[Y]^\times = \bar{k}^\times$ and $\text{Pic}(\bar{Y}) = 0$. Suppose Y has an open subset which is isomorphic to a torus T . Let $W \subset Y$ be a closed subset of codimension at least 2. Then $Y \setminus W$ satisfies strong approximation off v_0 , where v_0 is a place of k .*

Proof. Since $T \subset Y$, we have the following exact sequence

$$(1.1) \quad 0 \rightarrow \hat{T} \xrightarrow{\text{div}} \text{Div}_{\overline{Y} \setminus \overline{T}}(\overline{Y}) \rightarrow \text{Pic}(\overline{Y}) \rightarrow \text{Pic}(\overline{T}).$$

Since $\text{Pic}(\overline{Y}) = 0$, we have $\hat{T} \cong \text{Div}_{\overline{Y} \setminus \overline{T}}(\overline{Y})$.

Since $\text{Div}_{\overline{Y} \setminus \overline{T}}(\overline{Y})$ is a permutation Γ_k -module, there is an isomorphism $\hat{T} \cong \prod_i \mathbb{Z}[K_i/k]$, here all K_i/k are finite field extensions. Therefore

$$(1.2) \quad T \cong \prod_i R_{K_i/k}(\mathbb{G}_{m, K_i}).$$

Obviously we have the natural embedding $T \subset \prod_i R_{K_i/k}(\mathbb{A}_{K_i}^1) \cong \mathbb{A}_k^n$, here $n = \text{rank}(\text{Div}_{\overline{Y} \setminus \overline{T}}(\overline{Y}))$. Denote $Y' := \prod_i R_{K_i/k}(\mathbb{A}_{K_i}^1)$. In the following we will show:

there is a closed subset $W' \subset Y'$ of codimension at least 2 and $Y' \setminus W' \supset T \setminus W$, such that $T \setminus W \rightarrow Y \setminus W$ can be extended to $f : Y' \setminus W' \rightarrow Y \setminus W$.

We only need to show such W' exists over \overline{k} . Indeed, if we have found such $f : \overline{Y'} \setminus W' \rightarrow \overline{Y} \setminus \overline{W}$ over \overline{k} , then we also get the twisted map $f^\sigma : \overline{Y'} \setminus \sigma(W') \rightarrow \overline{Y} \setminus \overline{W}$ for any $\sigma \in \Gamma_k$. For $\sigma_1, \sigma_2 \in \Gamma_k$, the two maps f^{σ_1} and f^{σ_2} restriction to their intersection

$$(\overline{Y'} \setminus \sigma_1(W')) \cap (\overline{Y'} \setminus \sigma_2(W')) = \overline{Y'} \setminus (\sigma_1(W') \cup \sigma_2(W'))$$

are same by the reduced and separated property of $\overline{Y'}$ and $\overline{Y} \setminus \overline{W}$ over \overline{k} . Then we can replace W' by the union $\cup_{\sigma \in \Gamma_k} \sigma(W')$ which is in fact a finite intersection and has also codimension at least 2. Therefore we get a morphism $Y' \setminus W' \rightarrow Y \setminus W$ over k and W' has codimension at least 2.

Now we may assume $k = \overline{k}$. Then

$$T \cong \mathbb{G}_m^n = \text{Spec}(k[x_1^\pm, \dots, x_n^\pm]) \text{ and } Y' \cong \mathbb{A}_k^n = \text{Spec}(k[x_1, \dots, x_n]),$$

and the embedding $T \subset Y'$ induces the canonical embedding $\mathbb{G}_m^n \subset \mathbb{A}_k^n$. Let \tilde{W} be the closure of $T \cap W$ in Y' (maybe an empty set). Since $W \subset Y$ has codimension at least 2, $W \cap T \subset T$ has codimension at least 2. Therefore, $\tilde{W} \subset Y'$ has also codimension at least 2, hence all prime divisors of $Y' \setminus \tilde{W}$ in $Y' \setminus (T \cup \tilde{W})$ are the divisors defined by $x_i = 0$ with $i = 1, \dots, n$,

Denote $D_i := \text{div}(x_i)$ for $i = 1, \dots, n$, which are all prime divisors of $Y \setminus T$ by (1.1) and (1.2). Then $D_1 \setminus W, \dots, D_n \setminus W$ be all prime divisor in $Y \setminus (T \cup W)$ since $W \subset Y$ has codimension at least 2. Let p_i be the point of height one in $Y \setminus W$ associated to $D_i \setminus W$. Choose an open affine neighborhood of p_i in $Y \setminus W$ which is denoted by $\text{Spec}(A)$. Since the function field $k(Y \setminus W) = k(T) = k(x_1, \dots, x_n)$, we may assume that A is a subring of $k(x_1, \dots, x_n)$. For any $g \in k(x_1, \dots, x_n)^\times$, let $\text{div}'(g)$ be the principal divisor of g . We can see $\text{div}'(x_i) = (D_i \setminus W)$ since $\text{div}(x_i) = D_i$. Then x_i is a uniformizer of A_{p_i} . For any $g \in A \subset k(x_1, \dots, x_n)$, hence g can be written by the form $x_i^{n_i} g_1/g_2$ with $n_i \geq 0$, and $g_1, g_2 \in k[x_1, \dots, x_n]$ are relatively prime to x_i . Therefore we have the natural embedding $A \rightarrow k[x_1, \dots, x_n]_{(x_i)}$. Since $k[x_1, \dots, x_n]_{(x_i)}$ is the local ring of $Y' \setminus \tilde{W}$ at the point (x_i) of height one, we can extend $T \setminus W \rightarrow Y \setminus W$ to the point (of height one) (x_i) in $Y' \setminus \tilde{W}$. Since the divisors defined by $x_i = 0$ with $i = 1, \dots, n$ are all prime divisors of $Y' \setminus \tilde{W}$ in $Y' \setminus (T \cup \tilde{W})$, there is a closed subset $Z \subset Y' \setminus \tilde{W}$ of codimension at least 2, we can extend the map $T \setminus W \rightarrow Y \setminus W$ to $Y' \setminus (Z \cup \tilde{W}) \rightarrow Y \setminus W$. Let $W' = Z \cup \tilde{W}$. Let Z^c be the closure of Z in Y' , then $Z^c \setminus Z$ is contained in \tilde{W} . Hence $W' = Z^c \cup \tilde{W}$ is a closed subset of Y of codimension at least 2. We complete the proof of the claim.

Finally we prove that $Y \setminus W$ satisfies strong approximation off v_0 . For a given point $(p_v) \in (Y \setminus W)(\mathbf{A}_k)$, since $T \setminus W$ is open and dense in $Y \setminus W$, we can find a point $(p'_v) \in (T \setminus W)(\mathbf{A}_k)$ which is arbitrarily close to (p_v) . Since $Y' \cong \mathbf{A}_k^n$ and $W' \subset Y'$ has codimension at least 2, $Y' \setminus W'$ satisfies strong approximation off v_0 by Lemma 1.1. Since $(T \setminus W) \subset (Y' \setminus W')$, we can find a point $p' \in (Y' \setminus W')(k)$ is very close to $(p'_v)_{v \neq v_0}$ with respect to the adelic topology $(Y' \setminus W')(\mathbf{A}_k^{v_0})$. The image of p' by the map $f' : (Y' \setminus W') \rightarrow Y$ gives a k -point in $Y \setminus W$ which is very close to $(p_v)_{v \neq v_0}$ with respect to the adelic topology $(Y \setminus W)(\mathbf{A}_k^{v_0})$. \square

The following proposition implies the case that $\text{Pic}(\overline{X})$ is torsion free in Theorem 0.1.

Proposition 1.3. *Let X be a smooth and geometrically integral variety over k with $\overline{k}[X]^\times = \overline{k}^\times$, and X contains an open subset U which is a principal homogeneous space of a torus T . Let $W \subset X$ be a closed subset of codimension at least 2. If $\overline{k}[\mathcal{T}]^\times = \overline{k}^\times$ and $\text{Pic}(\overline{\mathcal{T}}) = 0$ for any universal torsor \mathcal{T} of X , then $X \setminus W$ satisfies strong approximation with algebraic Brauer–Manin obstruction off v_0 , where v_0 is a place of k . In particular, if $\text{Pic}(\overline{X})$ is torsion-free, then $X \setminus W$ satisfies strong approximation with algebraic Brauer–Manin obstruction off v_0 .*

Proof. We can assume $(X \setminus W)(\mathbf{A}_k)^{\text{Br}_1} \neq \emptyset$, hence universal torsors of $X \setminus W$ (or X) exist by [CTS87] or [Sko01, Proposition 6.1.4]. We will apply Proposition 1.2 to prove this proposition.

Let U be the open subset of X which is a principal homogeneous space of a torus T . We can see $\text{Pic}(\overline{U}) = 0$ since $\overline{U} \cong_{\overline{k}} \mathbb{G}_{m,\overline{k}}^n$ for some n . Since $\overline{k}[X]^\times = \overline{k}^\times$, we have the natural exact sequence

$$0 \rightarrow \overline{k}^\times \rightarrow \overline{k}[\overline{U}]^\times \rightarrow \text{Div}_{\overline{X} \setminus \overline{U}}(\overline{X}) \rightarrow \text{Pic}(\overline{X}) \rightarrow \text{Pic}(\overline{U}) = 0.$$

Let $\widehat{M} := \text{Div}_{\overline{X} \setminus \overline{U}}(\overline{X})$ and M the dual torus of \widehat{M} . By [Sko01, Lemma 2.4.3], we have the canonical Γ_k -isomorphism $\overline{k}[U]^\times / \overline{k}^\times \cong \widehat{T}$. We now want to apply [CTS87, Theorem 2.3.1, Corollary 2.3.4] for the local description of universal torsors over X . The restriction \mathcal{T}_U of the universal torsor \mathcal{T} to U has the form $\mathcal{T}_U = M \times_T U$, where the map $\psi : U \rightarrow T$ is given by a Γ_k -splitting of the following exact sequence

$$0 \rightarrow \overline{k}^\times \rightarrow \overline{k}[U]^\times \rightarrow \overline{k}[U]^\times / \overline{k}^\times (\cong \widehat{T}) \rightarrow 0.$$

It is clear that ψ is a k -isomorphism (e.g., [Sko01, Lemma 2.4.3]). Therefore $\mathcal{T}_U = M \times_T U \cong M$ is a (quasi-split) torus, then \mathcal{T} contains an dense open subset which is a torus (geometrically integral), hence \mathcal{T} is geometrically integral. Since X is smooth over k , \mathcal{T} is smooth over k .

On the other hand, if $\text{Pic}(\overline{X})$ is torsion free (and finitely generated), by [CTS87, Proposition 2.1.1], we have $\overline{k}[\mathcal{T}]^\times = \overline{k}^\times$ and $\text{Pic}(\overline{\mathcal{T}}) = 0$.

Since W has codimension at least 2, then

$$\overline{k}[Y \setminus W]^\times = \overline{k}[Y]^\times = \overline{k}^\times \text{ and } \text{Pic}(\overline{Y} \setminus \overline{W}) \cong \text{Pic}(\overline{Y}).$$

Let S be a group of multiplicative type, we have the following commutative diagram

$$(1.3) \quad \begin{array}{ccccccc} 0 & \longrightarrow & H^1(k, S) & \longrightarrow & H^1(X, S) & \xrightarrow{\text{type}} & \text{Hom}(\widehat{S}, \text{Pic}(\overline{X})) \\ & & \parallel & & \downarrow \text{Res} & & \downarrow \cong \\ 0 & \longrightarrow & H^1(k, S) & \longrightarrow & H^1(X \setminus W, S) & \xrightarrow{\text{type}} & \text{Hom}(\widehat{S}, \text{Pic}(\overline{X} \setminus \overline{W})). \end{array}$$

Therefore, any universal torsor \mathcal{T}' of $X \setminus W$ is the restriction of a universal torsor $f: \mathcal{T} \rightarrow X$ to $X \setminus W$, hence $\mathcal{T}' = \mathcal{T} \setminus f^{-1}(W) \subset \mathcal{T}$. Since f is flat and $W \subset X$ has codimension at least 2, $f^{-1}(W) \subset \mathcal{T}$ has also codimension at least 2. Therefore \mathcal{T}' satisfies strong approximation off v_0 by Proposition 1.2.

For a given point $(p_v) \in (X \setminus W)(\mathbf{A}_k)^{\text{Br}_1}$, by descent theory (see [CTS87], [Sko99, Theorem 3]), there is a universal torsor $f': \mathcal{T}' \rightarrow X \setminus W$ and $(r_v) \in \mathcal{T}'(\mathbf{A}_k)$ such that $(f'(r_v)) = (p_v)$.

By Proposition 1.2, we can find a point $r \in \mathcal{T}'(k)$ which can approximate $(r_v)_{v \neq v_0}$ arbitrary well in $\mathcal{T}'(\mathbf{A}_k^{\{v_0\}})$, then the point $p := f'(r)$ is very close to $(p_v)_{v \neq v_0}$ in $(X \setminus W)(\mathbf{A}_k^{\{v_0\}})$. \square

The above proposition dealt with the case $\text{Pic}(\overline{X})$ is torsion free. If $\text{Pic}(\overline{X})$ is not torsion free, the following lemma show the torsors of X under the maximal torsion group of $\text{Pic}(\overline{X})$ has the free Picard group, which ensures that the torsors satisfy strong approximation with algebraic Brauer–Manin obstruction by Proposition 1.3.

Lemma 1.4. *Let X be a smooth and geometrically integral variety over k with $\overline{k}[X]^\times = \overline{k}^\times$, and X contains an open subset U which is a principal homogeneous space of a torus T . Let \widehat{S} be the maximal torsion subgroup of $\text{Pic}(\overline{X})$ and λ the embedding $\widehat{S} \hookrightarrow \text{Pic}(\overline{X})$. Let \mathcal{T}' be a torsor of X of type λ . Then \mathcal{T}' is geometrically integral, $\overline{k}[\mathcal{T}']^\times = \overline{k}^\times$, $\text{Pic}(\overline{\mathcal{T}'})$ is torsion free, and the restriction \mathcal{T}'_U of \mathcal{T}' to U is a principal homogeneous space of a torus.*

Proof. Since $\overline{k}[X]^\times = \overline{k}^\times$, we have the following natural exact sequence

$$0 \rightarrow \widehat{T} (\cong \overline{k}[U]^\times / \overline{k}^\times) \rightarrow \text{Div}_{\overline{X} \setminus \overline{U}}(\overline{X}) \rightarrow \text{Pic}(\overline{X}) \rightarrow 0.$$

Let \widehat{M} be the inverse image of $\widehat{S} \subset \text{Pic}(\overline{X})$ in $\text{Div}_{\overline{X} \setminus \overline{U}}(\overline{X})$, \widehat{R} the inverse image of \widehat{M} in \widehat{T} . Therefore the following diagram is commutative

$$(1.4) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \widehat{R} & \xrightarrow{\gamma} & \widehat{M} & \longrightarrow & \widehat{S} \longrightarrow 0 \\ & & \downarrow \alpha & & \downarrow \beta & & \downarrow \lambda \\ 0 & \longrightarrow & \widehat{T} & \longrightarrow & \text{Div}_{\overline{X} \setminus \overline{U}}(\overline{X}) & \longrightarrow & \text{Pic}(\overline{X}) \longrightarrow 0, \end{array}$$

where α, β and γ are natural embedding, $\text{coker}(\alpha)$ and $\text{coker}(\beta)$ are torsion free. Let R (resp. M) be the dual torus of \widehat{R} (resp. \widehat{M}). By the local description of torsors ([CTS87, Theorem 2.3.1, Corollary 2.3.4]), we have $\mathcal{T}'_U = M \times_R U$ which is a principal homogenous space of a torus $M \times_R T$, since U is a principal homogeneous space of T . Since \mathcal{T}'_U is geometrically integral, \mathcal{T}' is also geometrically integral.

In the following we will show $\overline{k}[\mathcal{T}']^\times = \overline{k}^\times$ and $\text{Pic}(\overline{\mathcal{T}'})$ is torsion-free.

We may assume $k = \bar{k}$. Hence we may assume $U = T$. By the local description of torsors ([CTS87, Theorem 2.3.1, Corollary 2.3.4]),

$$(1.5) \quad \mathcal{T}'_U = M \times_R U = M \times_R T,$$

where the map $T \rightarrow R$ is induced by a morphism $\hat{R} \rightarrow \bar{k}[T]^\times$, which lifts $\alpha : \hat{R} \rightarrow \bar{k}[T]^\times / \bar{k}^\times$. Over \bar{k} , the universal torsor is unique, therefore we can choose the morphism $T \rightarrow R$ in (1.5) is the induced morphism of tori by α in (1.4). Since $\text{coker}(\alpha)$ in (1.4) is torsion free, we can write $\hat{T} \cong \hat{R} \oplus \hat{R}'$, where \hat{R}' is torsion free with the dual torus R' . Then $T \cong R \times R'$. Hence

$$\mathcal{T}'_U \cong M \times_R (R \times R') \cong M \times R'.$$

The natural morphism $\mathcal{T}'_U \cong M \times R' \rightarrow T$ is a group morphism of tori, which is given by the morphism

$$i : \hat{T} \cong \hat{R} \oplus \hat{R}' \rightarrow \widehat{\mathcal{T}'_U} \cong \hat{M} \oplus \hat{R}', (a, a') \mapsto (\gamma(a), a').$$

Then

$$(1.6) \quad \text{coker}(i) = \text{coker}(\gamma) \cong \hat{S}.$$

Define $\text{div} : \bar{k}(X)^\times \rightarrow \text{Div}(\bar{X})$ by $v \mapsto \text{div}(v)$, similarly for $\text{div}' : \bar{k}(\mathcal{T}')^\times \rightarrow \text{Div}(\overline{\mathcal{T}'})$. By (1.6), $\text{coker}(i)$ is a torsion group. Therefore, for any $u \in \bar{k}[\mathcal{T}']^\times \subset \bar{k}[\mathcal{T}'_U]^\times$, there is a positive integer n such that u^n is an image of $v \in \bar{k}[T]^\times$ up to an element in \bar{k}^\times . Since $u \in \bar{k}[\mathcal{T}']^\times$, we have $\text{div}'(u) = 0$. Therefore

$$j(\text{div}(v)) = \text{div}'(u^n) = 0,$$

where $j : \text{Div}(\bar{X}) \rightarrow \text{Div}(\overline{\mathcal{T}'})$ is induced by the structure map $f : \mathcal{T}' \rightarrow X$. Since f is flat and surjective, j is injective. This implies $\text{div}(v) = 0$, hence $v \in \bar{k}[X]^\times = \bar{k}^\times$. Then $u^n \in \bar{k}^\times$, hence $u \in \bar{k}^\times$. Therefore $\bar{k}[\mathcal{T}']^\times = \bar{k}^\times$.

We have the following commutative diagram

$$(1.7) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \hat{T}(\cong \bar{k}[U]^\times / \bar{k}^\times) & \longrightarrow & \text{Div}_{\bar{X} \setminus \bar{U}}(\bar{X}) & \longrightarrow & \text{Pic}(\bar{X}) \longrightarrow 0 \\ & & \downarrow i & & \downarrow j' & & \downarrow s \\ 0 & \longrightarrow & \widehat{\mathcal{T}'_U}(\cong \bar{k}[\mathcal{T}'_U]^\times / \bar{k}) & \longrightarrow & \text{Div}_{\overline{\mathcal{T}' \setminus \mathcal{T}'_U}}(\overline{\mathcal{T}'}) & \longrightarrow & \text{Pic}(\overline{\mathcal{T}'}) \longrightarrow 0, \end{array}$$

where $j' : \text{Div}_{\bar{X} \setminus \bar{U}}(\bar{X}) \rightarrow \text{Div}_{\overline{\mathcal{T}' \setminus \mathcal{T}'_U}}(\overline{\mathcal{T}'})$ is the restriction of j to $\text{Div}_{\bar{X} \setminus \bar{U}}(\bar{X})$. Since $\mathcal{T}' \rightarrow X$ is finite étale, for any prime divisor D of X , $j(D)$ is a sum of different prime divisors (without multiples). Therefore $\text{coker}(j')$ is torsion free. Applying the snake lemma to (1.7), we have the following exact sequence

$$0 \rightarrow \ker(s) \rightarrow \text{coker}(i) \xrightarrow{\phi_1} \text{coker}(j') \xrightarrow{\phi_2} \text{coker}(s) \rightarrow 0.$$

Since $\text{coker}(i) \cong \hat{S}$ is a torsion group by (1.6) and $\text{coker}(j')$ is torsion free, ϕ_1 is a zero map. Therefore $\ker(s) \cong \text{coker}(i) \cong \hat{S}$, and $\text{coker}(s) \cong \text{coker}(j')$ is torsion free. Note that $\text{Pic}(\bar{X})/\ker(s) = \text{Pic}(\bar{X})/\hat{S}$ is also torsion free, hence $\text{Pic}(\mathcal{T}')$ is torsion free. \square

The proof of Theorem 0.1. If $\text{Pic}(\bar{X})$ is torsion free, the proof follows from Proposition 1.3.

Since $W \subset X$ has codimension 2, then $\text{Pic}(\bar{X}) \cong \text{Pic}(\bar{X} \setminus \bar{W})$. Let \hat{S} be the maximal torsion subgroup of $\text{Pic}(\bar{X})$. Let λ be the natural embedding $\hat{S} \rightarrow \text{Pic}(\bar{X})$

and λ' the induced embedding $\widehat{S} \rightarrow \text{Pic}(\overline{X} \setminus \overline{W})$. Since $W \subset X$ has codimension at least 2, by diagram (1.3), any torsor of $X \setminus W$ of type λ' is the restriction of a torsor of X of type λ . Let $f' : \mathcal{T}' \rightarrow X \setminus W$ be the torsor of type λ' , which is the restriction of the torsor $f : \mathcal{T} \rightarrow X$ of type λ , hence $\mathcal{T}' = \mathcal{T} \setminus f^{-1}(W)$. Since $W \subset X$ has codimension at least 2, $f^{-1}(W) \subset \mathcal{T}$ has also codimension at least 2. By Proposition 1.3 and Lemma 1.4, \mathcal{T}' satisfies strong approximation with algebraic Brauer–Manin obstruction off v_0 .

For a given point

$$(p_v) \in (X \setminus W)(\mathbf{A}_k)^{\text{ét}, \text{Br}_1} \subset \bigcup_{\text{type}(f': \mathcal{T}' \rightarrow X \setminus W) = \lambda'} f'(\mathcal{T}'(\mathbf{A}_k)^{\text{Br}_1}),$$

then $(p_v) = (f'(r_v))$ for some $f' : \mathcal{T}' \rightarrow X \setminus W$ and $(r_v) \in \mathcal{T}'(\mathbf{A}_k)^{\text{Br}_1}$. Since \mathcal{T}' satisfies strong approximation with algebraic Brauer–Manin obstruction off v_0 , we can find a point $r \in \mathcal{T}'(k)$ which can approximate $(r_v)_{v \neq v_0}$ arbitrary well in $\mathcal{T}'(\mathbf{A}_k^{\{v_0\}})$, then the point $p := f'(r)$ is very close to $(p_v)_{v \neq v_0}$ in $(X \setminus W)(\mathbf{A}_k^{\{v_0\}})$. \square

Remark 1.5. (1) Harari and Skorobogatov (see [HS13]) had extended the descent theory to arbitrary smooth algebraic varieties without the condition $\overline{k}[X]^\times = \overline{k}^\times$. Possibly we can apply the generalized descent theory to deal with some cases without this condition. For some special case, e.g., toric varieties, without the conditions $\overline{k}[X]^\times = \overline{k}^\times$, the similar strong approximation version had been proved by Cao and Xu in [CX14].

- (2) If $\overline{k}[X]^\times \neq \overline{k}^\times$, generally $X \setminus W$ does not satisfy strong approximation with Brauer–Manin obstruction off v_0 even when X satisfies strong approximation with Brauer–Manin obstruction off v_0 . For example, $k = \mathbb{Q}$, $\mathbb{G}_a \times \mathbb{G}_m$ satisfies strong approximation with Brauer–Manin obstruction off the infinite place, but $X = \mathbb{G}_a \times \mathbb{G}_m \setminus \{(0, 1)\}$ does not (see [CX14, Example 5.2]).
- (3) If $\text{Pic}(\overline{X})$ is free, then $\text{Br}_1(X)/\text{Br}_0(X) \cong H^1(k, \text{Pic}(\overline{X}))$ is finite. Otherwise, generally $\text{Br}_1(X)/\text{Br}_0(X)$ is not finite, e.g., $X \subset \mathbb{A}_k^{n+1}$ is the smooth locus of the affine variety (as in Proposition 3.1) defined by $cx_1 \cdots x_n = y^2$ with $n \geq 2$ and $c \in k^\times$, then $\text{Pic}(\overline{X}) = (\mathbb{Z}/2\mathbb{Z})^{n-1}$ is not torsion free, hence $\text{Br}_1(X)/\text{Br}_0(X) \cong H^1(k, \mathbb{Z}/2\mathbb{Z})^{n-1}$ is infinite.

Let X be as in Theorem 0.1. If $\text{Pic}(\overline{X})$ is not torsion free, for a universal torsor \mathcal{T} of X , it seems that $\text{Pic}(\overline{\mathcal{T}}) \neq 0$ generally. Maybe there is Brauer–manin obstruction to strong approximation on \mathcal{T} . Therefore, sometimes strong approximation with Brauer–Manin obstruction does not hold on X off one place. In the following, we will give such an example, which is similar to [CTW12, Example 5.10].

Example 1.6. Let p, q be primes with $p \equiv 3 \pmod{4}$, $q \equiv 1 \pmod{8}$, $\left(\frac{p}{q}\right) = 1$ and $\left(\frac{p}{q}\right)_4 = -1$ (e.g. $(p, q) = (19, 17)$). Let $f(x, y, z) := p(qx + y)y + qz^2$. Let $\mathcal{D} \subset \mathbb{P}_{\mathbb{Z}}^2$ be the closed subset defined by $f(x, y, z) = 0$. Let $\mathcal{V} := \mathbb{P}_{\mathbb{Z}}^2 \setminus \mathcal{D}$ and $V = \mathcal{V} \otimes_{\mathbb{Z}} \mathbb{Q}$. Then $\overline{\mathbb{Q}}[V]^\times = \overline{\mathbb{Q}}^\times$, $\text{Pic}(\overline{V}) \cong \mathbb{Z}/2\mathbb{Z}$, and there is an open subset of V which is isomorphic to the torus \mathbb{G}_m^2 , and

$$\left(V(\mathbb{R}) \times \prod_{l < \infty} \mathcal{V}(\mathbb{Z}_l) \right)^{\text{Br}(V)} \neq \emptyset, \text{ but } \mathcal{V}(\mathbb{Z}) = \emptyset.$$

Proof. Since $V = \mathbb{P}_{\mathbb{Q}}^2 \setminus D$ with $D := \mathcal{D} \otimes_{\mathbb{Z}} \mathbb{Q}$ is a hypersurface of $\mathbb{P}_{\mathbb{Q}}^2$ of degree 2, we have the following exact sequence

$$0 \rightarrow \overline{\mathbb{Q}}[V]^{\times} / \overline{\mathbb{Q}}^{\times} \xrightarrow{\text{div}} \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} (\cong \text{Pic}(\mathbb{P}_{\mathbb{Q}}^2)) \rightarrow \text{Pic}(\overline{V}) \rightarrow 0,$$

this implies $\overline{\mathbb{Q}}[V]^{\times} = \overline{\mathbb{Q}}^{\times}$ and $\text{Pic}(\overline{V}) \cong \mathbb{Z}/2\mathbb{Z}$.

Let $\mathbb{A}_{\mathbb{Q}}^2$ be the open subset of $\mathbb{P}_{\mathbb{Q}}^2$ defined by $y \neq 0$. Then $\mathbb{A}_{\mathbb{Q}}^2 \cap V \subset \mathbb{A}_{\mathbb{Q}}^2$ is defined by $p(qx+1) + qz^2 \neq 0$. We can define an isomorphism $\mathbb{A}_{\mathbb{Q}}^2 \rightarrow \mathbb{A}_{\mathbb{Q}}^2$ by $(x, z) \mapsto (p(qx+1) + qz^2, z)$, then the open subset $\mathbb{A}_{\mathbb{Q}}^2 \cap V$ of V is isomorphic to $\mathbb{A}_{\mathbb{Q}}^2 \setminus \{x=0\}$. Obviously $\mathbb{A}_{\mathbb{Q}}^2 \setminus \{x=0\}$ contains an open subset \mathbb{G}_m^2 , hence V contains an open subset which is isomorphic to the torus \mathbb{G}_m^2 .

Since the units of \mathbb{Z} are just ± 1 , the integral point (x, y, z) of $\mathcal{V}(\mathbb{Z}) \subset \mathbb{P}_{\mathbb{Z}}^2(\mathbb{Z})$ is defined by $f(x, y, z) = \pm 1$. Obviously $f(x, y, z) = -1$ is not solvable over \mathbb{Z} since it is not solvable over \mathbb{Z}_p , note that $\left(\frac{-q}{p}\right) = -1$. Let $\mathcal{U} \subset \mathbb{A}_{\mathbb{Z}}^3$ be the affine variety defined by $f(x, y, z) = 1$, there is a natural map

$$\mathcal{U} \rightarrow \mathcal{V}, (x, y, z) \mapsto (x : y : z).$$

Then $\mathcal{V}(\mathbb{Z}) = \emptyset$ follows from $\mathcal{U}(\mathbb{Z}) = \emptyset$. Let $U := \mathcal{U} \otimes_{\mathbb{Z}} \mathbb{Q}$. We know $\text{Br}(U)/\text{Br}(\mathbb{Q}) = \mathbb{Z}/2\mathbb{Z}$ and it is generated by the quaternion $A := (y, q)$ (see [CTX09]).

Obviously $U(\mathbb{R}) \neq \emptyset$. In the following we will show $\mathcal{U}(\mathbb{Z}_l) \neq \emptyset$ for all l :

- $l \nmid pq$, we can choose $(x, y, z) = (p^{-1}q^{-1} - q^{-1}, 1, 0) \in \mathbb{Z}_l^3$;
- $l = p$, we can choose $(x, y, z) = (0, 0, \sqrt{q^{-1}}) \in \mathbb{Z}_l^3$;
- $l = q$, we can choose $(x, y, z) = (0, \sqrt{p^{-1}}, 0) \in \mathbb{Z}_l^3$.

Let $P_l := (x_l, y_l, z_l)$ be a point in $\mathcal{U}(\mathbb{Z}_l)$ for l a finite prime and $P_{\infty} \in U(\mathbb{R})$ for $l = \infty$. We will show the Brauer–Manin set of \mathcal{U} is empty:

- If $l = \infty$, since $q > 0$, we have $A(P_{\infty}) = 1$;
- If $l = 2$, since $q \equiv 1 \pmod{8}$, obviously $A(P_2) = (y, q)_2 = 1$;
- If $l \nmid 2q$ and $\left(\frac{q}{l}\right) = 1$, then q is a square in \mathbb{Z}_l , obviously $A(P_l) = (y_l, q)_l = 1$;
- If $l \nmid 2q$ and $\left(\frac{q}{l}\right) = -1$, then $l \nmid y_l$ by the equation $f(x_l, y_l, z_l) = 1$, hence y_l is a unit in \mathbb{Z}_l . Therefore

$$A(P_l) = (y_l, q)_l = 1;$$

- If $l = q$, we have $py_l^2 \equiv 1 \pmod{q}$, then y_l is a unit and

$$A(P_l) = \left(\frac{y_l}{q}\right) = \left(\frac{y_l^2}{q}\right)_4 = \left(\frac{p}{q}\right)_4^{-1} = -1.$$

Therefore

$$\left(U(\mathbb{R}) \times \prod_{l < \infty} \mathcal{U}(\mathbb{Z}_l) \right)^{\text{Br}(U)} = \emptyset,$$

this implies $\mathcal{U}(\mathbb{Z}) = \emptyset$, hence $\mathcal{V}(\mathbb{Z}) = \emptyset$.

Let $X \subset \mathbb{P}_{\mathbb{Q}}^3$ be defined by $f(x, y, z) = t^2$. Then $U \subset X$ is the open subset defined by $t \neq 0$. We can define a morphism $\pi : X \rightarrow \mathbb{P}_{\mathbb{Q}}^2$ by $(x : y : z : t) \mapsto (x : y : z)$ which is ramified of degree 2 over D and is a $\mathbb{Z}/2\mathbb{Z}$ -torsor outside D . By [CTW12, Sequence (5.6)], $\text{Br}(V)/\text{Br}(\mathbb{Q})$ is 2-torsion. By [CTW12, Diagram (5.7)], we

have $\pi^*(\text{Br}(V)) \subset \text{Br}(X) = \text{Br}(\mathbb{Q})$, therefore the image in $V(\mathbb{R}) \times \prod_{l < \infty} \mathcal{V}(\mathbb{Z}_l)$ of $U(\mathbb{R}) \times \prod_{l < \infty} \mathcal{U}(\mathbb{Z}_l)$ is orthogonal to $\text{Br}(V)$, this implies

$$\left(V(\mathbb{R}) \times \prod_{l < \infty} \mathcal{V}(\mathbb{Z}_l) \right)^{\text{Br}(V)} \neq \emptyset. \quad \square$$

2. TORIC VARIETIES

In this section, we will prove Theorem 0.2. In particular, if $\text{Pic}(\overline{X})$ is torsion free, in fact this result follows from Theorem 0.1. However, sometimes the Picard groups of some toric varieties are not torsion free (see [Cox11, Example 4.2.3]). We will give a single proof for the general case, the key idea of this proof is from [DW13, Lemma 3.1].

The proof of Theorem 0.2. We may assume $(X \setminus W)(\mathbf{A}_k)^{\text{Br}_1} \neq \emptyset$. Since Brauer–Manin obstruction is only one to the Hasse principle and weak approximation for toric varieties (e.g. [Sko01, Theorem 6.3.1]), then we may assume that X contains an open subset which is isomorphic to the torus T .

Let \hat{T} be the group of characters of T . Let $N := \hat{T}^* = \text{Hom}_{\mathbb{Z}}(\hat{T}, \mathbb{Z})$, Δ a fan in $N_{\mathbb{R}} := N \otimes \mathbb{R}$ and $X = X_{\Delta}$ the toric variety associated to Δ . Since $\overline{k}[X]^{\times} = \overline{k}^{\times}$, Δ can span $N_{\mathbb{R}}$.

Let $\text{Div}_{\overline{T}}(\overline{X})$ be the \overline{T} -invariant Weil divisor of \overline{X} . By [Cox11, Theorem 4.1.3], we have the following exact sequence

$$(2.1) \quad 0 \rightarrow \hat{T}(\cong \overline{k}[T]^{\times} / \overline{k}^{\times}) \xrightarrow{\text{div}} \text{Div}_{\overline{T}}(\overline{X}) \rightarrow \text{Pic}(\overline{X}) \rightarrow 0.$$

Let $f : \mathcal{T} \rightarrow X$ be a universal torsor of X . Let $\hat{M} := \text{Div}_{\overline{T}}(\overline{X})$ and M the dual torus of \hat{M} . Let $i : M \rightarrow T$ be the induced morphism by $\text{div} : \hat{T} \rightarrow \hat{M}$. We now want to apply [CTS87, Theorem 2.3.1, Corollary 2.3.4] for the local description of universal torsors of X . The restriction \mathcal{T}_T of the universal torsor \mathcal{T} to T has the form $\mathcal{T}_T = M \times_T T$, which is the pull-back of $i : M \rightarrow T$ and $\sigma : T \rightarrow T$, here σ is a translation of T induced by a splitting of the following exact sequence

$$0 \rightarrow \overline{k}^{\times} \rightarrow \overline{k}[T]^{\times} \rightarrow \overline{k}[T]^{\times} / \overline{k}^{\times} \rightarrow 0.$$

We have the following commutative diagram

$$\begin{array}{ccc} M \times_T T & \xrightarrow{p_1} & M \\ \downarrow p_2 & & \downarrow i \\ T & \xleftarrow{\sigma} & T, \end{array}$$

where p_1 and p_2 are the natural projections. Since p_1 is an isomorphism, the universal torsor $\mathcal{T}_T \cong M$ with the structure morphism $\sigma \circ i : M \rightarrow T$.

Similarly by the commutative diagram (1.3), since $W \subset X$ has codimension 2, any universal torsor \mathcal{T}' of $X \setminus W$ is the restriction of a universal torsor $f : \mathcal{T} \rightarrow X$ to $X \setminus W$, then $\mathcal{T}' = \mathcal{T} \setminus f^{-1}(W) \subset \mathcal{T}$ and the restriction of \mathcal{T}' to $T \setminus W$ is $M \setminus (\sigma \circ i)^{-1}(W)$.

Since \hat{M} is a permutation Γ_k -module, there are finite field extensions K_1, \dots, K_n over k such that $\hat{M} \cong \prod_{i=1}^n \mathbb{Z}[K_i/k]$. Therefore

$$M \cong \prod_{i=1}^n \mathbf{R}_{K_i/k}(\mathbb{G}_{m, K_i}).$$

Obviously we have the natural embedding $M(\cong \mathcal{T}_T) \subset \prod_i \mathbf{R}_{K_i/k}(\mathbf{A}_{K_i}^1)(\cong \mathbf{A}_k^{n'})$ for some n'). Denote $Y := \prod_i \mathbf{R}_{K_i/k}(\mathbf{A}_{K_i}^1)$. We claim:

there is a closed subvariety $Z \subset Y$ of codimension at least 2 and $(Y \setminus Z) \supset (M \setminus (\sigma \circ i)^{-1}(W))$, such that the restricted map of $\sigma \circ i$ on $M \setminus (\sigma \circ i)^{-1}(W)$ can be extended to $Y \setminus Z \rightarrow X \setminus W$.

By Lemma 1.1, $Y \setminus Z$ satisfies strong approximation off v_0 . Therefore, by [DW13, Lemma 3.1], $X \setminus W$ satisfies strong approximation with algebraic Brauer–Manin obstruction off v_0 . Now we prove this claim.

Since X is a toric variety which has a T -action, the translation $\sigma : T \rightarrow T$ can be extended to the unique isomorphism of X which we also denote by σ . Obviously $W' := \sigma^{-1}(W) \subset X$ has also codimension at least 2. Therefore we only need to show that such $Z \subset Y$ exists such that $i : M \setminus i^{-1}(W') \rightarrow T \setminus W'$ can be extended to $Y \setminus Z \rightarrow X \setminus W'$.

With similar arguments as in the 4-th paragraph in the proof of Proposition 1.2, we only need to prove this claim over \bar{k} . Now we assume $k = \bar{k}$. Denote $\bar{\Delta}$ to be the fan of Δ omitting the Γ_k -action.

We shall call a 1-dimensional cone a ray. Let $\bar{\Delta}(1)$ be the set of rays of $\bar{\Delta}$. Let D_ρ be the \bar{T} -invariant Weil divisor associated to $\rho \in \bar{\Delta}(1)$. Recall $\hat{M} = \sum_{\rho \in \bar{\Delta}(1)} \mathbb{Z} D_\rho$. Denote $\tilde{N} := \hat{M}^* = \text{Hom}_{\mathbb{Z}}(\hat{M}, \mathbb{Z})$, $\tilde{D}_\rho \in \tilde{N}$ the dual of D_ρ , and $\tilde{N}_{\mathbb{R}} := \tilde{N} \otimes_{\mathbb{Z}} \mathbb{R}$. It is clear that $\{\tilde{D}_\rho : \rho \in \bar{\Delta}(1)\}$ is a basis of $\tilde{N}_{\mathbb{R}}$.

Let C be the fan associated to the simplicial cone in $\tilde{N}_{\mathbb{R}}$ generated by $\{\tilde{D}_\rho : \rho \in \bar{\Delta}(1)\}$. Then $\bar{Y} \cong Y_C$, the toric variety of C with the natural \bar{M} -action. Let $R_\rho \subset \tilde{N}_{\mathbb{R}}$ be the ray generated by \tilde{D}_ρ . Then $\{R_\rho : \rho \in \bar{\Delta}(1)\}$ are all rays of C . Let

$$C' := \{0\} \cup \{R_\rho : \rho \in \bar{\Delta}(1)\} \subset C,$$

which is a subfan of C . Denote $Y_{C'} \subset Y_C$ to be the open toric subvariety of the fan C' . Similarly, let

$$\bar{\Delta}' := \{0\} \cup \bar{\Delta}(1) \subset C.$$

Denote $X_{\bar{\Delta}'} \subset X_{\bar{\Delta}}$ to be the open toric subvariety of the fan $\bar{\Delta}'$.

We will prove the claim by two steps. First we will show such Z exists for X (i.e., $W = \emptyset$), then we get a morphism $g : Y \setminus Z \rightarrow X$ which extends $i : M \rightarrow T$. Secondly, we will show $g^{-1}(W') \subset Y \setminus Z$ has codimension at least 2, then the restriction of g to $Y \setminus (Z \cup g^{-1}(W'))$ extends the morphism $M \setminus i^{-1}(W') \rightarrow T \setminus W'$, it is clear that $(Z \cup g^{-1}(W')) \subset Y$ is a closed subset of codimension at least 2.

Step 1: we will prove that such Z exists for X . In fact, we will prove:

there is a toric morphism $g : Y_{C'} \rightarrow X_{\bar{\Delta}'}$ (compatible with $\bar{M} \rightarrow \bar{T}$), then we had extended $\bar{M} \rightarrow \bar{T}$ to $Y_{C'} \rightarrow \bar{X}$.

Note that $Y_{C'} \subset Y_C$ and $Y_C \setminus Y_{C'}$ has codimension 2 since C' contains all rays in C .

Let $\tilde{g} : \tilde{N}_{\mathbb{R}} \rightarrow N_{\mathbb{R}}$ be the morphism induced by $\text{div} : \hat{T} \rightarrow \hat{M}$. The claim follows from that there is a morphism of fans $C' \rightarrow \bar{\Delta}'$ which is induced by $\tilde{g} : \tilde{N}_{\mathbb{R}} \rightarrow N_{\mathbb{R}}$.

For any $\rho \in \bar{\Delta}(1)$, let $u_\rho \in N$ be the minimal generator of ρ . By [Cox11, Proposition 4.1.2], we have

$$\text{div} : \hat{T} \rightarrow \hat{M}, \chi \mapsto \text{div}(\chi) = \sum_{\rho \in \bar{\Delta}} \chi(u_\rho) D_\rho.$$

For $\sigma \in \bar{\Delta}(1)$, $\tilde{g}(\tilde{D}_\sigma) \in N(= \text{Hom}_{\mathbb{Z}}(\hat{T}, \mathbb{Z}))$ and

$$\tilde{g}(\tilde{D}_\sigma) = \left(\chi \mapsto \tilde{D}_\sigma \left(\sum_{\rho \in \bar{\Delta}(1)} \chi(u_\rho) D_\rho \right) = \chi(u_\sigma) \right).$$

Since the natural isomorphism $N \cong N^{**}$ is given by $x \mapsto (\chi \mapsto \chi(x))$, we have $f(\tilde{D}_\sigma) = u_\sigma$. Therefore $\tilde{g}(R_\rho) = \rho$ for any $\rho \in \bar{\Delta}(1)$, hence \tilde{g} induces a morphism of fans $C' \rightarrow \bar{\Delta}'$. Then we complete the proof of the first step.

Step 2: we will prove that $g^{-1}(W) \subset Y_{C'}$ is a close subset of codimension ≥ 2 .

Let U_{R_ρ} (resp. V_ρ) be the toric open subvariety of $Y_{C'}$ (resp. $X_{\Delta'}$) associated to $R_\rho \in C'$ (resp. $D_\rho \in \bar{\Delta}(1)$). Let $g_\rho : U_{R_\rho} \rightarrow V_\rho$ be the induced toric morphism by $\tilde{g} : \tilde{N} \rightarrow N$ since $\tilde{g}(R_\rho) = \rho$. We have $Y_{C'}$ (resp. $X_{\Delta'}$) is the union of U_{R_ρ} (resp. V_ρ), $\rho \in \bar{\Delta}(1)$. Then g is in fact glued by these g_ρ . Therefore, we only need to show that $g_\rho^{-1}(W') \subset U_{R_\rho}$ has codimension at least 2.

We may write $\hat{M} \cong \hat{M}_1 \oplus \hat{M}_2$, \hat{M}_1 and \hat{M}_2 free abelian groups, such that $\hat{T} \xrightarrow{\text{div}} \hat{M}$ is identity with the composite map

$$(2.2) \quad \hat{T} \xrightarrow{d} \hat{M}_1 \xrightarrow{j} \hat{M}_1 \oplus \hat{M}_2,$$

here the cokernel of d is finite and j is the natural embedding by $j(m) = (m, 0)$ for any $m \in \hat{M}_1$. Let $N' := \hat{M}_1^* = \text{Hom}(\hat{M}_1, \mathbb{Z})$ be the dual of \hat{M}_1 , obviously $\tilde{N} \cong N' \oplus \hat{M}_2^*$. Therefore we have the dual morphism

$$\tilde{N} \cong N' \oplus \hat{M}_2^* \xrightarrow{j^*} N' \xrightarrow{d^*} N,$$

here j^* is in fact the natural projection. Let $R'_\rho \subset N'_\mathbb{R} := N' \otimes_{\mathbb{Z}} \mathbb{R}$ be the ray of the image of R_ρ by $j^* \otimes \mathbb{R}$. Let $U'_{R'_\rho}$ be the toric variety associated to $R'_\rho \subset N'_\mathbb{R}$. Obviously $U_{R_\rho} \cong U'_{R'_\rho} \times \mathbb{G}_m^m$ where $m = \dim(\hat{M}_2)$, and d^* induces a toric morphism $g'_\rho : U'_{R'_\rho} \rightarrow V_\rho$. Then g_ρ is identity with the composite map

$$U'_{R'_\rho} \times \mathbb{G}_m^m \xrightarrow{p_1} U'_{R'_\rho} \xrightarrow{g'_\rho} V_\rho,$$

where p_1 is the natural projection.

Let $\rho^\vee \subset \hat{T} \otimes_{\mathbb{Z}} \mathbb{R}$ (resp. $R_\rho^\vee \subset \hat{M}_1 \otimes_{\mathbb{Z}} \mathbb{R}$) be the dual cone of ρ (resp. R_ρ). Then the morphism d in (2.2) induces a morphism $d_\rho : k[\rho^\vee \cap \hat{T}] \rightarrow k[R_\rho^\vee \cap \hat{M}_1]$. Since the cokernel of d is finite, there exists $m > 0$ such that $m\rho' \in d(\rho^\vee \cap \hat{T})$ for any $\rho' \in R_\rho^\vee \cap \hat{M}_1$. Therefore, d_ρ is a finite and injective morphism, i.e., g'_ρ is a finite and surjective morphism. Since $W' \cap V_\rho \subset V_\rho$ has codimension at least 2, $g'^{-1}_\rho(W' \cap V_\rho) \subset U'_{R'_\rho}$ has also codimension at least 2. Therefore, $g^{-1}_\rho(W' \cap V_\rho) \subset U_\rho$ has codimension at least 2 since $g_\rho = g'_\rho \circ p_1$ and p_1 is a projection. \square

Remark 2.1. (1) Let N be a lattice and $N_\mathbb{R} = N \otimes_{\mathbb{Z}} \mathbb{R}$. Let Δ be a fan in $N_\mathbb{R}$, $X = X_\Delta$ the toric variety of Δ . In fact, the condition $\bar{k}[X]^\times = \bar{k}^\times$ is equivalent to that Δ can span $N_\mathbb{R}$ over \mathbb{R} .

(2) If W is not empty, the condition $\bar{k}[X]^\times = \bar{k}^\times$ is necessary. For example, let $k = \mathbb{Q}$, $X = \mathbb{G}_a \times \mathbb{G}_m$ a toric variety with $\bar{k}[X]^\times / \bar{k}^\times \cong \mathbb{Z}$, $W = \{(0, 1)\} \subset X$, $X \setminus W$ does not satisfy strong approximation with Brauer–Manin obstruction off the infinite place (see [CX14, Example 5.2]).

3. SOME EXAMPLES

In this section, we will give some typical varieties which satisfy strong approximation with algebraic Brauer–Manin obstruction off v_0 , which is not totally from Theorem 0.1.

Proposition 3.1. *Let $m, n \geq 1$ and let K_i, L_j be finite field extensions over k , where $i = 1, \dots, m$ and $j = 1, \dots, n$. Let X be the smooth locus of the affine variety defined by*

$$(3.1) \quad \prod_{i=1}^m N_{K_i/k}(\mathbf{w}_i) = c \prod_{j=1}^n N_{L_j/k}(\mathbf{z}_j)^{s_j},$$

with $c \in k^\times$ and $s_j \geq 1$, $j = 1, \dots, n$. Then $\bar{k}[X]^\times = \bar{k}^\times$ and X satisfies strong approximation with algebraic Brauer–Manin obstruction off v_0 .

Proof. It is clear that X contains an open subset U by $\prod_{i=1}^m N_{K_i/k}(\mathbf{w}_i) \neq 0$, which is a principal homogeneous space of the torus defined by

$$\prod_{i=1}^m N_{K_i/k}(\mathbf{w}'_i) \cdot \prod_{j=1}^n N_{L_j/k}(\mathbf{z}'_j)^{s_j} = 1.$$

Over \bar{k} , X can be viewed as the smooth locus of the affine variety defined by

$$(3.2) \quad w_1 \cdots w_{m'} = c z_1^{r_1} \cdots z_{n'}^{r_{n'}}$$

with all $r_j \geq 1$. Let $D_{i,j}$ be the divisor of X defined by $w_i = z_j = 0$ for i, j . We can see $\text{div}(w_i) = \sum_{j=1}^{n'} r_j D_{i,j}$ and $\text{div}(z_j) = \sum_{i=1}^{m'} D_{i,j}$.

Now we will show $\bar{k}[X]^\times = \bar{k}^\times$. Since every $f \in \bar{k}[X]^\times$ has the form

$$f = c' w_1^{\alpha_1} \cdots w_{m'}^{\alpha_{m'}} z_1^{\beta_1} \cdots z_{n'}^{\beta_{n'}}$$

with $c' \in \bar{k}^\times$ and $\alpha_1, \dots, \alpha_{m'}, \beta_1, \dots, \beta_{n'} \in \mathbb{Z}$, and

$$0 = \text{div}(f) = \sum_{i=1}^{m'} \sum_{j=1}^{n'} (\alpha_i r_j + \beta_j) D_{i,j},$$

we have $\beta_j = -\alpha_i r_j$ for i, j , therefore $\alpha_1 = \cdots = \alpha_{m'}$ and $\beta_j = -\alpha_1 r_j$ for all j , so f is constant by (3.2).

If $\gcd(s_1, \dots, s_n) = 1$, then $\gcd(r_1, \dots, r_{n'}) = 1$, hence $\text{Pic}(\bar{X})$ is torsion free by an easy computation. Then strong approximation with algebraic Brauer–Manin obstruction off v_0 follows from Theorem 0.1.

For the general case, here we provide a single proof. As in the proof of Proposition 1.3, for any universal torsor \mathcal{T} , the restriction \mathcal{T}_U is a quasi-split torus, and $\mathcal{T}_U \cong \prod_{i,j} \mathbf{R}_{K_i \otimes_k L_j/k}(\mathbb{G}_m)$, and the map $\mathcal{T}_U \rightarrow U$ is given by

$$(u_{ij})_{i,j} \mapsto (\eta_i \prod_j N_{K_i \otimes_k L_j/K_i}(u_{ij})^{s_j})_i \times (\xi_j \prod_i N_{K_i \otimes_k L_j/L_j}(u_{ij}))_j$$

with $\eta_i \in K_i^\times$, $\xi_j \in L_j^\times$ such that $\prod_i N_{K_i/k}(\eta_i) = c \prod_j N_{L_j/k}(\xi_j)^{s_j}$. Obviously \mathcal{T}_U can naturally embed into

$$Y := \prod_{i,j} \mathbf{R}_{K_i \otimes_k L_j/k}(\mathbb{A}^1) \cong \mathbb{A}_k^m$$

for some m , hence $\mathcal{T}_U \rightarrow X$ can be extended to $g : Y \rightarrow X'$, where X' is the affine variety defined by (3.1). By the equation (3.2), the singular locus V of X' , i.e., $X' \setminus X$ is contained in W which has codimension 2, where W is the union of

$$\{w_i = w_j = 0, z_l = 0\} \text{ with } i \neq j.$$

Therefore, the map $\mathcal{T}_U \rightarrow U$ can be extended to $Y \setminus f^{-1}(V) \rightarrow X$, and $f^{-1}(V) \subset f^{-1}(W)$ has codimension at least 2. By Lemma 1.1, $Y \setminus f^{-1}(V)$ satisfies strong approximation off v_0 . By [DW13, Lemma 3.1], X satisfies strong approximation with algebraic Brauer–Manin obstruction off v_0 . \square

Corollary 3.2. *Let L and K be finite Galois extensions over k . Let X be the smooth locus of the affine variety over k defined by*

$$N_{L/k}(\mathbf{w}) = cN_{K/k}(\mathbf{z})$$

with $c \in k^\times$, and let T be the multi-norm 1 torus over k defined by $N_{L/k}(\mathbf{w}')N_{K/k}(\mathbf{z}') = 1$. Then

$$\mathrm{Br}_1(X)/\mathrm{Br}_0(X) \cong H^2(L.K/k, \hat{T}).$$

Furthermore, if $L \cap K = k$ and the degrees of the maximal abelian subextensions of L/k and K/k are relatively prime, then $\mathrm{Br}_1(X) = \mathrm{Br}_0(X)$ and X satisfies strong approximation off v_0 .

Proof. By Proposition 3.1, $\bar{k}[X]^\times = \bar{k}^\times$. Let U be the open subset of X by $N_{L/k}(\mathbf{w}) = cN_{K/k}(\mathbf{z}) \neq 0$, then U is a principal homogeneous space of T and $\bar{k}[U]^\times / \bar{k}^\times \cong \hat{T}$ as Γ_k -modules, hence we have the following exact sequence

$$(3.3) \quad 0 \rightarrow \hat{T} \rightarrow \mathrm{Div}_{\bar{X} \setminus \bar{U}}(\bar{X}) \rightarrow \mathrm{Pic}(\bar{X}) \rightarrow \mathrm{Pic}(\bar{U}) = 0.$$

Let $G_1 := \mathrm{Gal}(L/k)$, $G_2 := \mathrm{Gal}(K/k)$ and $G := \mathrm{Gal}(L.K/k)$. Since $\mathrm{Div}_{\bar{X} \setminus \bar{U}}(\bar{X}) \cong \mathbb{Z}[G_1] \otimes \mathbb{Z}[G_2]$ as Γ_k -module and split by $L.K$, then we can view $\mathrm{Div}_{\bar{X} \setminus \bar{U}}(\bar{X})$ as a G -module. For $i \geq 1$, by Shapiro's Lemma, we have

$$H^i(G, \mathrm{Div}_{\bar{X} \setminus \bar{U}}(\bar{X})) = H^i(L.K/K, \mathbb{Z}[G_1]) = H^i(L/L \cap K, \mathbb{Z}[G_1]) = 0.$$

The sequence (3.3) implies the following exact sequence

$$H^1(G, \mathrm{Div}_{\bar{X} \setminus \bar{U}}(\bar{X})) \rightarrow H^1(G, \mathrm{Pic}(X_{L.K})) \rightarrow H^2(G, \hat{T}) \rightarrow H^2(G, \mathrm{Div}_{\bar{X} \setminus \bar{U}}(\bar{X})),$$

which implies

$$H^1(G, \mathrm{Pic}(X_{L.K})) \cong H^2(G, \hat{T}).$$

Since $\mathrm{Pic}(\bar{X})$ is free and split by $L.K$, we have

$$\mathrm{Br}_1(X)/\mathrm{Br}_0(X) \cong H^1(k, \mathrm{Pic}(\bar{X})) \cong H^1(G, \mathrm{Pic}(X_{L.K})) \cong H^2(G, \hat{T}).$$

We have the following exact sequence

$$0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Z}[G_1] \oplus \mathbb{Z}[G_2] \rightarrow \hat{T} \rightarrow 0.$$

Hence

$$(3.4) \quad \begin{aligned} H^2(G, \mathbb{Z}) &\xrightarrow{i} H^2(L.K/L, \mathbb{Z}) \oplus H^2(L.K/K, \mathbb{Z}) \rightarrow H^2(G, \hat{T}) \\ &\rightarrow H^3(G, \mathbb{Z}) \xrightarrow{j} H^3(L.K/L, \mathbb{Z}) \times H^3(L.K/K, \mathbb{Z}), \end{aligned}$$

where the maps i and j are induced by the restriction maps. If $L \cap K = k$, we have $\text{Gal}(L.K/K) \cong G_1$, $\text{Gal}(L.K/L) \cong G_2$ and $G \cong G_1 \times G_2$ canonically, it is clear that i is surjective. On the other hand,

$$H^1(G_1, H^1(G_2, \mathbb{Q}/\mathbb{Z})) = \text{Hom}(G_1^{ab}, \text{Hom}(G_2^{ab}, \mathbb{Q}/\mathbb{Z})),$$

where G_1^{ab} and G_2^{ab} are their maximal abelian quotient. By our assumption, $\#G_1^{ab}$ and $\#G_2^{ab}$ are relatively prime, hence

$$H^1(G_1, H^1(G_2, \mathbb{Q}/\mathbb{Z})) = 0.$$

By the Künneth formula ([NSW99, Exercise II.1.7]), the map j in (3.4) is injective, hence (3.4) implies $H^2(G, \hat{T}) = 0$. Therefore

$$\text{Br}_1(X) = \text{Br}_0(X).$$

By Proposition 3.1, X satisfies strong approximation off v_0 . \square

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D. WEI: ACADEMY OF MATHEMATICS AND SYSTEM SCIENCE, CAS, BEIJING
100190, P.R.CHINA

E-mail: `dshwei@amss.ac.cn`